THE KING'S SCHOOL

2005
Higher School Certificate
Trial Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-10
- All questions are of equal value

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Mathematics

Question	Algebra and Number	Geometry	Functions	Trigonometry	Differential Calculus	Integral Calculus	Total
1	(a), (c), (d), (e), (f)			(b)			12
2			(a)		(b)		12
3	(d)	(a)				(b), (c)	12
4				(a)		(b), (c)	12
5	(b)(ii)(iii)(iv)		(a)		(b)(i)		12
6	(b)		(a)(i)(ii)		(a)(iii)		12
7		(b)			(a)		12
8			(a)(i)		(b)ii)	(a)(ii), (b)(i)(iii)	12
9	(a)		(b)(iv)(v)	(b)(ii)(iii)	(b)(i)		12
10			(a), (b)(iii)	(b)(i)(ii)	(b)(iv)(v)		12
Marks	27	10	26	15	25	17	120

Total marks – 120
Attempt Questions 1-10
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Convert 2005 degrees to radians, correct to the nearest integer. **1**

(b) State the amplitude and period of the function $y = -\sin 2x$ **2**

(c) Simplify $\frac{2}{3} - \frac{3x+8}{12}$ **2**

(d) Express $\frac{\sqrt{2}+2}{\sqrt{2}}$ in simplest form. **2**

(e) Find $\sum_{n=1}^{3} \frac{6}{n}$ **1**

(f) Solve

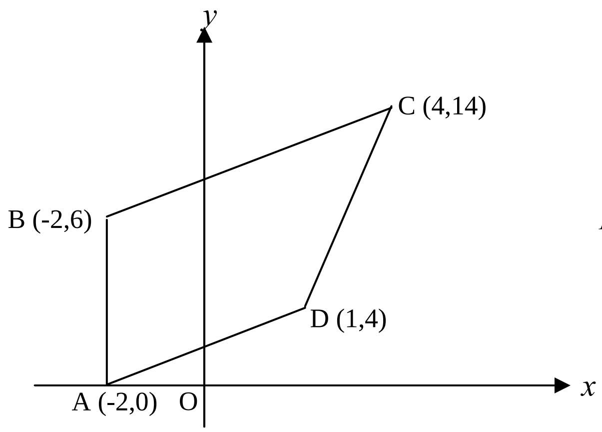
(i) $|x-3| < 9$ **2**

(ii) $2005^x = 21$, correct to 1 decimal place. **2**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



Not to scale.

ABCD is a quadrilateral with vertices A(-2,0), B(-2,6), C(4,14) and D(1,4).

(i) Show that AD is parallel to BC. **2**

(ii) Find the sum of the lengths of AD and BC. **2**

(iii) Show that the equation of the line AD is $3y - 4x - 8 = 0$ **2**

(iv) Hence, or otherwise, find the area of the quadrilateral ABCD. **3**

(b) Differentiate

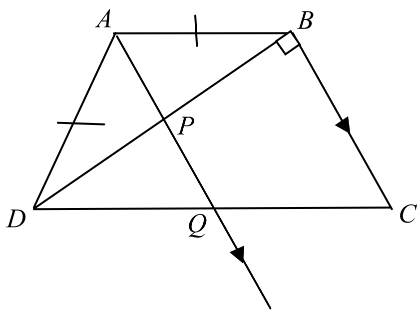
(i) $(2x+1)^{-1}$ **1**

(ii) $\dfrac{x}{x^2+1}$ **2**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



In the diagram, $AB = AD$ and $\angle DBC = 90°$.

Line APQ is parallel to line BC and meets DB at P and DC at Q.

(i) Give a reason why $\angle APB = 90°$. **1**

(ii) Prove that ΔABP is congruent to ΔADP. **2**

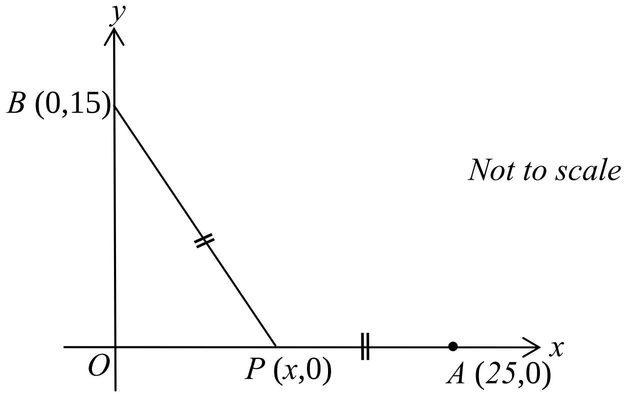
(iii) Deduce that $DQ = QC$. **2**

(b) Find a primitive function of $\dfrac{x^3}{x^4+1}$ **1**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{8}} 2\sec^2 2x \quad dx$ **2**

Question 3 continues on next page

(d)



$P(x,0)$ is the point between $O(0,0)$ and $A(25,0)$ such that $PA = PB$, where $B = (0,15)$.

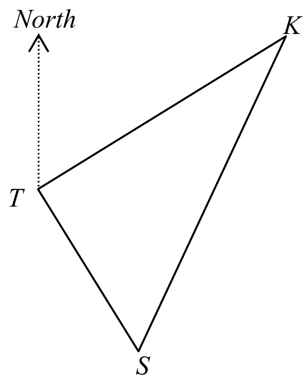
(i) Explain why $PB = 25 - x$. **1**

(ii) Find the value of x. **3**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The bearing of K from T is $070°$ and $TK = 400\text{km}$.

The bearing of S from T is $150°$ and $TS = 213\text{km}$.

(i) Show that $\angle KTS = 80°$. **1**

(ii) Find KS, nearest km. **2**

(iii) Use the sine rule to find the bearing of S from K, nearest degree. **3**

(b) Use Simpson's Rule once to approximate $\int_1^2 x \sin x \quad dx$,
correct to 2 decimal places. **3**

(c) The curve $y = f(x)$ has a stationary point at $x = 1$ and $f''(x) = 6x - 2$.

Find the x coordinate at which there is another stationary point. **3**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The quadratic equation $Q(x) = 2x^2 + (k-3)x + (k+3) = 0$ has no real roots.

(i) Prove that $k^2 - 14k - 15 < 0$ **2**

(ii) Find the values of k. **2**

(iii) For what values of k is $Q(x) > 0$ for all real values of x? **1**

(b) The population P of a country town on January 1, 1995 was 1730 and on January 1, 2005 was 1160. The town's population is known to be changing according to the equation $\frac{dP}{dt} = kP$ where t is time in years measured from January 1, 1995 and k is a constant.

(i) Verify that $P = Ae^{kt}$, where A is a constant, satisfies the equation $\frac{dP}{dt} = kP$. **2**

(ii) State the value of A **1**

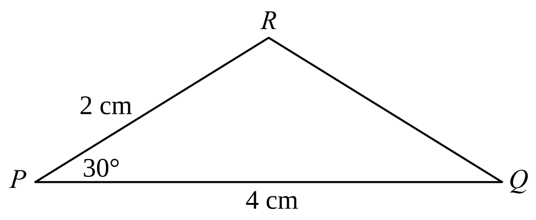
(iii) Find the value of k, correct to 1 significant figure. **2**

(iv) Approximately how many people are expected to leave the town during the year 2005? **2**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



ΔPQR has a fixed angle $P = 30°$ but side PR is increasing at 4cm/min and side PQ at 2cm/min.

Initially, $PR = 2\text{cm}$ and $PQ = 4\text{cm}$.

Let $A(t)$ be the area of ΔPQR after t minutes.

(i) Show that $A(t) = 2t^2 + 5t + 2$ **2**

(ii) When is the area of ΔPQR equal to 102cm^2? **2**

(iii) At what rate is the area of ΔPQR changing after 10 minutes? **2**

Question 6 continues next page

(b) George borrows \$20 000 from Saint Bank. This loan plus interest is to be repaid in equal monthly instalments of \$399 over five years. Interest of 7.2% p.a. is compounded monthly on the balance owing at the start of each month.

Let $\$A_n$ be the amount owing after n months.

(i) Over the five year repayment period, how much interest is charged? **1**

(ii) Show that $A_1 = 19721$. **1**

(iii) Clearly show that $A_2 = 20\,000 \times 1.006^2 - 399(1 + 1.006)$. **1**

(iv) Deduce that $A_n = 66\,500 - 46\,500 \times 1.006^n$. **2**

(v) After two years of repayments George decides on the very next day to repay the loan in full by one payment.

How much will this one payment be? **1**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

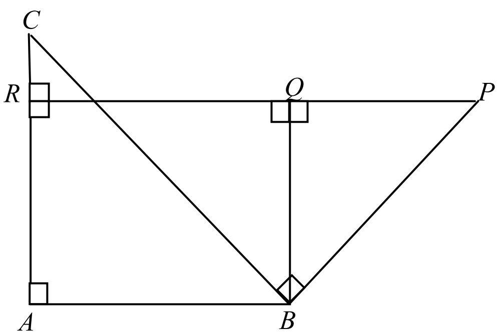
(a) Consider the function $f(x) = 8x^3 - 3x^4$.

(i) For what values of x is the function decreasing? **2**

(ii) Show that the point (0,0) is a point of inflection. **2**

(iii) Sketch the function showing x intercepts and stationary points. **3**

(b)



In the diagram there are six right angles marked.

(i) Prove that ΔABC is similar to ΔBPQ. **2**

(ii) If further, $BC = BP$, which test proves that ΔABC is congruent to ΔBPQ? **1**

(iii) Deduce that $PR = AB + AC$. **2**

End of Question 7

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

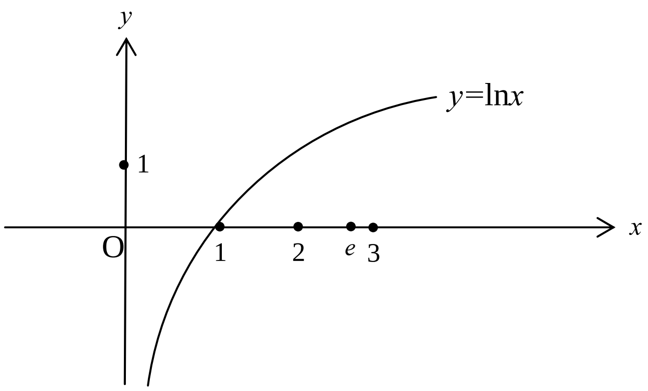
(a) The line $y = 3 - 2x$ and the parabola $y = 2x - x^2$ meet at the points (1,1) and (3,-3).

[Do Not Show This]

(i) On the same diagram sketch the line and the parabola and include on your diagram the x intercepts for each. **2**

(ii) Find the area of the region bounded by the line and the parabola. **3**

(b)



(i) The region bounded by the curve $y = \ln x$ and the y axis between $y = 0$ and $y = 1$ is revolved about the y axis.

Find the volume of the solid of revolution. **3**

(ii) Show that $\frac{d}{dx}\left[x\left((\ln x)^2 - 2\ln x + 2 \right) \right] = (\ln x)^2$ **2**

(iii) The region bounded by the curve $y = \ln x$ and the x axis between $x = 1$ and $x = e$ is revolved about the x axis.

Find the volume of the solid of revolution. **2**

End of Question 8

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The sum of the first n terms of an arithmetic series is given by $S_n = n(n+a)$, where a is a constant.

Find the common difference. **2**

(b) A particle is moving along the x axis. Initially it is at the origin with velocity 1.

Its displacement at any time $t \geq 0$ is given by $x = \sin^2 t + \sin t$.

(i) Show that its velocity at any time $t \geq 0$ is given by $\dot{x} = \cos t(2\sin t + 1)$ **2**

(ii) Find the first five times that the particle stops. **3**

(iii) The acceleration of the particle can be expressed as $\ddot{x} = 2\cos 2t - \sin t$

[DO NOT SHOW THIS]

Find the displacement, velocity and acceleration when $t = \frac{\pi}{2}$ **2**

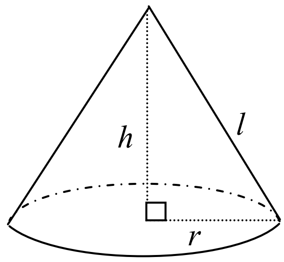
(iv) Use (iii) to explain why immediately after $t = \frac{\pi}{2}$ the particle will move towards the origin. **1**

(v) Find the least distance that the particle moves between its stopping positions. **2**

End of Question 9

Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)

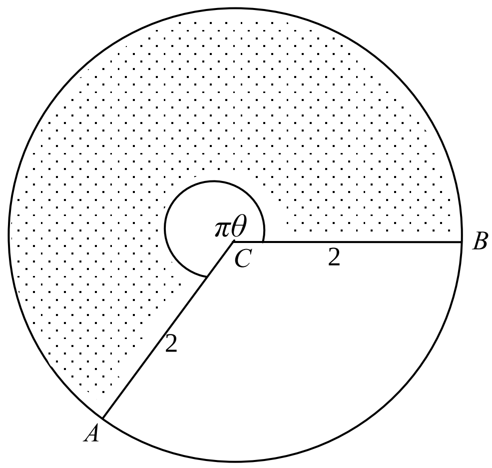


A cone has radius r, height h and slant height l.

The volume of the cone $V = \frac{\pi}{3} r^2 h$

Show that the volume of the cone can be expressed as $V = \frac{\pi}{3}\sqrt{l^2 r^4 - r^6}$. **2**

(b)



The angle at the centre C of a circle of radius 2cm is $\pi\theta$ radians, $0 < \theta < 2$, as shown in the diagram.

(i) Write down the length of the arc AB of this sector. **1**

(ii) This sector is cut from the circle along the radii CA and CB and folded to make a cone.

Find the radius of the cone. **1**

(iii) Show that the volume of the cone is given by $V = \dfrac{\pi}{3}\sqrt{4\theta^4 - \theta^6}$ **1**

(iv) Find the value of θ, correct to two decimal places, for which the volume of the cone is a maximum. **4**

(v) Sketch the graph of V against θ using the same scale on both axes. **3**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

TKS MATHEMATICS TRIAL H.S.C. 2005 SOLUTIONS

Qn 1

(a) $2005 \times \frac{\pi}{180}$ radians $= 35^c$

(b) amplitude $= 1$, period $= \frac{2\pi}{2} = \pi$

(c) $\frac{8-(3x+8)}{12} = -\frac{3x}{12} = -\frac{x}{4}$

(d) $\frac{\sqrt{2}+2}{\sqrt{2}} = 1 + \frac{2}{\sqrt{2}} = 1 + \sqrt{2}$

(e) $6 + 3 + 2 = 11$

(f) (i) $-9 < x-3 < 9$

$\therefore -6 < x < 12$

(ii) $x \ln 2005 = \ln 21$

$x = \frac{\ln 21}{\ln 2005} = 0.4$, 1 d.p.

Qu 2

(a) (i) gradient $AD = \frac{4}{1 - -2} = \frac{4}{3}$

$\text{grd } BC = \frac{14-6}{4 - -2} = \frac{8}{6} = \frac{4}{3}$

$\therefore AD \parallel BC$ (equal grds)

(ii) $AD + BC = \sqrt{3^2+4^2} + \sqrt{6^2+8^2}$

$= 5 + 10 = 15$

(iii) AD is $y = \frac{4}{3}(x - -2)$

or $3y = 4x + 8$

i.e. $3y - 4x - 8 = 0$

(iv) ◊ABCD is a trapezium, from (i)

⊥ distance from $B(-2, 6)$ to AD

$= \frac{18 + 8 - 8}{\sqrt{3^2+4^2}} = \frac{18}{5}$

$\therefore$ Area $= \frac{1}{2}(15)\frac{18}{5} = 27 \text{ u}^2$

(b) (i) $-(2x+1)^{-2} \cdot 2 = \frac{-2}{(2x+1)^2}$

(ii) $\frac{(x^2+1)1 - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

Qu 3

(a) (i) $\angle APB = \angle PBC$, alt $\angle$s in // lines

$= 90^\circ$, given

(ii) Since $\angle APB = 90^\circ$ then $\angle APD = 90^\circ$, $\angle DPB$ straight

$AD = AB$ (hypotenuses), given

AP is common

$\therefore \Delta$s congruent, RHS

(iii) $DP = PB$, corr. sides in cong Δs (ii)

$\therefore DQ = QC$, ratio intercept theorem in // lines

(b) $$\frac{1}{4}\int \frac{4x^3}{x^4+1}\,dx = \frac{1}{4}\ln(x^4+1)$$

(c) $$\int_0^{\frac{\pi}{8}} 2\sec^2 2x\,dx = \frac{2}{2}\Big[\tan 2x\Big]_0^{\frac{\pi}{8}}$$

$$= \tan\frac{\pi}{4} - 0$$

$$= 1$$

(d) (i) $PA = 25 - x$ $\therefore PB = 25 - x$ since $PA = PB$

(ii) In ΔOPB, $x^2 + 15^2 = (25-x)^2$

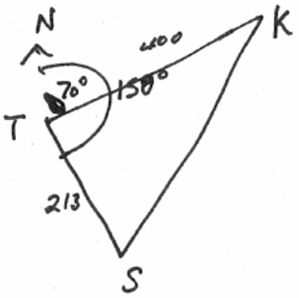
ie. $x^2 + 225 = 625 - 50x + x^2$

or $50x = 400$

$x = 8$

Q 4

(a) (i)



$\therefore \angle KTS = 150° - 70° = 80°$

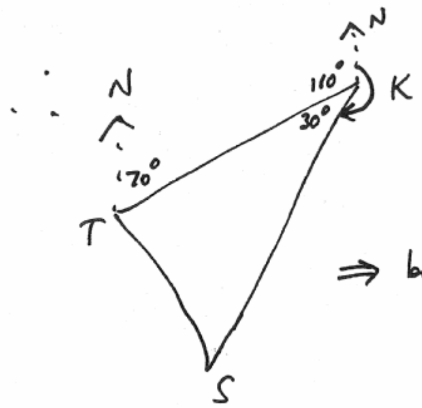
(ii) $KS^2 = 400^2 + 213^2 - 2 \times 400 \times 213 \cos 80°$

$\Rightarrow KS = 419$ km

(iii) $\dfrac{\sin K}{213} = \dfrac{\sin 80°}{419}$

$\therefore \sin K = \dfrac{213 \sin 80°}{419} \Rightarrow \hat{K} = 30°$, nearest degree

$\therefore$



$\Rightarrow$ bearing of S from K $= 360° - 140°$
$= 220°$

(b) $\displaystyle\int_1^2 x \sin x \, dx \approx \frac{1}{6} \cdot 1 \left[\sin 1 + 2 \sin 2 + 4 \cdot \tfrac{3}{2} \sin 1.5\right]$

≈ 1.44

(c) $\therefore f'(x) = 3x^2 - 2x + c \quad : \quad f'(1) = 3 - 2 + c = 0$
$c = -1$

$\therefore f'(x) = 3x^2 - 2x - 1 = (x-1)(3x+1)$

$\Rightarrow$ stat. pt when $3x + 1 = 0$ i.e. $x = -\dfrac{1}{3}$

Qn 5

(a) (i) We must have $\Delta < 0$

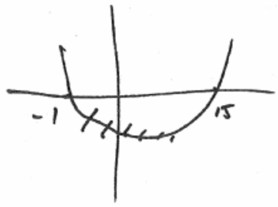
$$\therefore (k-3)^2 - 4 \times 2(k+3) < 0$$

i.e. $k^2 - 6k + 9 - 8k - 24 < 0$

or $k^2 - 14k - 15 < 0$

(ii) From (i), $(k+1)(k-15) < 0$

$$\Rightarrow -1 < k < 15$$



(iii) i.e. positive definite $\Rightarrow \Delta < 0$

$$\therefore -1 < k < 15$$

(b) (i) If $P = Ae^{kt}$ then $\frac{dP}{dt} = Ake^{kt}$

$$= k(Ae^{kt})$$

$$= kP$$

(ii) $t = 0$, $P = 1730 \Rightarrow A = 1730$

(iii) $t = 10$, $P = 1160 \Rightarrow 1160 = 1730e^{10k}$

$$\therefore e^{10k} = \frac{116}{173} \Rightarrow 10k = \ln\left(\frac{116}{173}\right)$$

i.e. $k = \frac{1}{10}\ln\left(\frac{116}{173}\right) = -0.04$, 1 sig. fig

(iv) Jan, 2006 $\Rightarrow t = 11$ $\therefore P = 1730e^{-0.04 \times 11} = 1730e^{-0.44}$

$$\simeq 1114 \quad (1110 \text{ will do})$$

$\therefore$ expect $1160 - 1114 = 46$ to leave (50 will do)

Qu 6

(a) (i) After t minutes, $PR = 2 + 4t$ and $PQ = 4 + 2t$

$$\therefore A(t) = \frac{1}{2}(2+4t)(4+2t)\sin 30^\circ$$
$$= \frac{1}{4}(2+4t)(4+2t)$$
$$= (1+2t)(2+t)$$
$$= 2t^2 + 5t + 2$$

(ii) $2t^2 + 5t + 2 = 102$

$$\Rightarrow 2t^2 + 5t - 100 = 0$$
$$\therefore t = \frac{-5 + \sqrt{25 + 800}}{4} \quad \text{since } t > 0$$
$$= \frac{-5 + \sqrt{825}}{4} \text{ min} \quad [\approx 5.93 \text{ min}]$$

(iii) $\frac{dA}{dt} = 4t + 5 = 4 \times 10 + 5$ after 10 minutes

$$= 45 \text{ cm}^2/\text{min}$$

(b) (i) Interest $= \$399 \times 12 \times 5 - \$20\,000 = \$3940$

(ii) $A_1 = 20000 \times 1.006 - 399 = 19721$

[* 7.2% p.a. $= \frac{0.072}{12}$ p. month $= 0.006$]

(iii) $A_2 = A_1(1.006) - 399$

$$= (20000 \times 1.006 - 399)1.006 - 399$$
$$= 20000 \times 1.006^2 - 399(1 + 1.006)$$

(iv) (iii) $\Rightarrow A_n = 20000 \times 1.006^n - 399(1 + 1.006 + \ldots + 1.006^{n-1})$

$$= 20000 \times 1.006^n - 399\left(\frac{1.006^n - 1}{.006}\right)$$
$$= 20000 \times 1.006^n - 66500(1.006^n - 1)$$
$$= 66500 - 46500 \times 1.006^n$$

(v) The one payment $= A_{24} = 66500 - 46500 \times 1.006^{24} \approx \$12821$

[or \$12820.99]

Qu 7

(a) (i) $f'(x) = 24x^2 - 12x^3$

$= 12x^2(2-x) < 0$ if $2 - x < 0$

ie $x > 2$

$\therefore$ decreasing for $x > 2$ [$x \geq 2$ will do]

(ii) $f''(x) = 48x - 36x^2 = 12x(4-3x)$

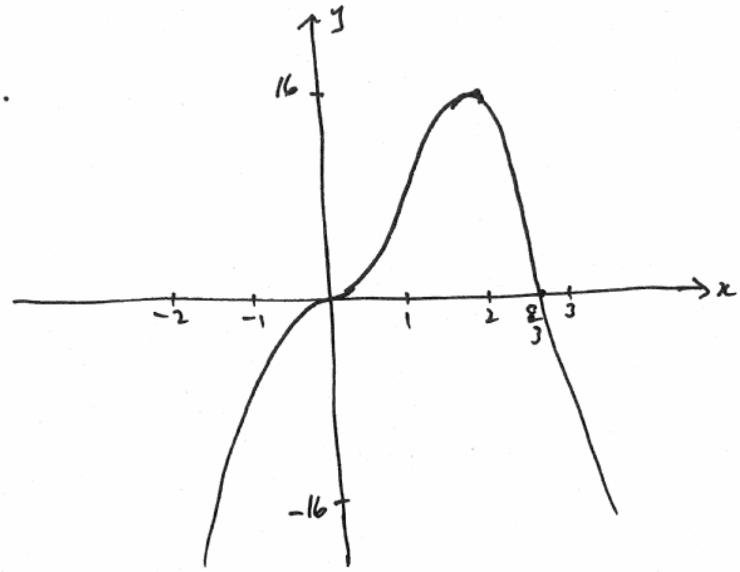
Now, $f''(0) = 0$ and $f''(-1) < 0$, $f''(1) > 0$ [change in concavity]

$\therefore$ $(0,0)$ is a pt of inflection

(iii) $f'(x) = 0 \Rightarrow x = 0, 2$ ie stat pts $(0,0)$, $(2,16)$

$y = 0, 16$

$f(x) = 0 \Rightarrow x^3(8-3x) = 0 \Rightarrow x$ intercepts $0, \frac{8}{3}$



(b) (i) (Lots of alternatives)

$\angle ABQ = 90°$, angle sum of $\square ABQR$

$\therefore \angle ABC = \angle PBQ$, both complements of $\angle CBQ$

and $\angle A = \angle Q = 90°$, given

$\therefore \triangle ABC \,|||\, \triangle BPQ$, 2 angles equal

(ii) AAS

(iii) $PR = PQ + QR$

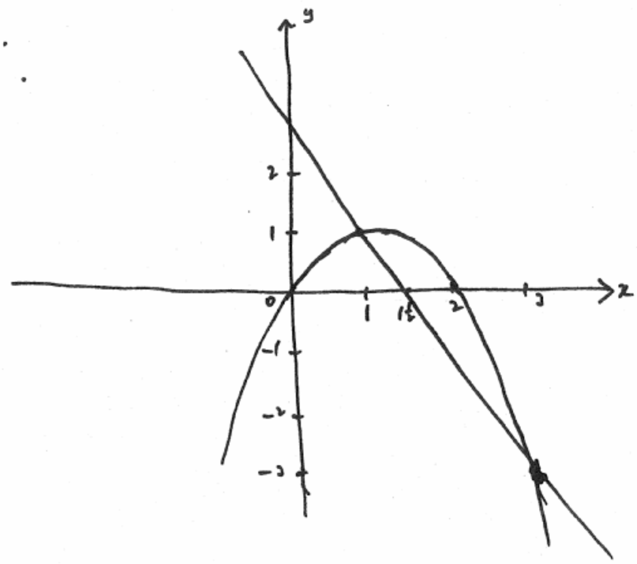
$= AC + QR$, cong $\triangle$s in (ii)

$= AC + AB$, opp sides of rectangle $ABQR$

Qu 8

(a) (i) For $y = 3-2x$, $y=0 \Rightarrow x = \frac{3}{2}$

For $y = 2x - x^2 = x(2-x)$, $y = 0 \Rightarrow x = 0, 2$



(ii) $A = \int_1^3 2x - x^2 - (3-2x)\, dx$

$= \int_1^3 4x - x^2 - 3 \, dx$

$= \left[2x^2 - \frac{x^3}{3} - 3x\right]_1^3$

$= 18 - 9 - 9 - \left(2 - \frac{1}{3} - 3\right) = \frac{4}{3} \ u^2$

(b) (i) $V = \pi \int_0^1 x^2 \, dy$ $\quad$: $y = \log_e x \Rightarrow x = e^y$

$\therefore x^2 = e^{2y}$

$\therefore V = \pi \int_0^1 e^{2y} \, dy = \frac{\pi}{2} \left[e^{2y} \right]_0^1$

$= \frac{\pi}{2} \left(e^2 - 1 \right) \quad u^3$

(ii) $\frac{d}{dx} x \left((\ln x)^2 - 2\ln x + 2 \right) = x \left(2\ln x \cdot \frac{1}{x} - \frac{2}{x} \right) + \left((\ln x)^2 - 2\ln x + 2 \right) 1$

$= 2\ln x - 2 + (\ln x)^2 - 2\ln x + 2$

$= (\ln x)^2$

(iii) $V = \pi \int_1^e y^2 \, dx = \pi \int_1^e (\ln x)^2 \, dx$

$= \pi \left[x \left((\ln x)^2 - 2\ln x + 2 \right) \right]_1^e$ $\quad$ from (ii)

$= \pi \left(e(1 - 2 + 2) - 1(0 - 0 + 2) \right)$

$= \pi (e - 2) \quad u^3$

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(a) $S_1 = T_1 = 1 + a$

$S_2 = 2(2+a) = 4 + 2a$

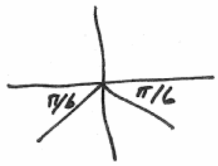
$\therefore T_2 = 4 + 2a - (1+a) = 3 + a$

$\therefore$ common difference $= 3 + a - (1+a) = 2$

(b) (i) $\dot{x} = 2\sin t \cos t + \cos t$

ie $\dot{x} = \cos t\,(2\sin t + 1)$

(ii) $\dot{x} = 0 \Rightarrow \cos t = 0$ or $\sin t = -\frac{1}{2}$



$\therefore t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

or $t = \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$

ie. first 5 times are $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, \frac{5\pi}{2}$

(iii) $t = \frac{\pi}{2}$, $x = 1 + 1 = 2$

$\dot{x} = 0$

$\ddot{x} = 2\cos\pi - 1 = -2 - 1 = -3$

(iv) Since $\ddot{x} = -3 < 0$ then $\dot{x}$ will decrease

ie $\dot{x}$ will become negative

$\Rightarrow$ particle moves from $x = 2$ towards the origin

(v) $t = \frac{7\pi}{6}$, $x = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$t = \frac{3\pi}{2}$, $x = 1 - 1 = 0$

$t = \frac{11\pi}{6}$, $x = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$t = \frac{5\pi}{2}$, $x = 1 + 1 = 2$

$\Rightarrow$ least distance $= \frac{1}{4}$

(v) $V = \frac{\pi}{3}\sqrt{4\theta^4 - \theta^6}$, $0 < \theta < 2$

$\theta \to 0$, $V \to 0$

$\theta \to 2$, $V \to 0$

$\theta = 1.63$, $V = \frac{\pi}{3}\sqrt{4(1.63)^4 - 1.63^6} \approx 3.22$

$[\theta = 1, V \approx 1.8]$